

Thermoelastic stress analysis

The thermoelastic effect

The relationship between mechanical deformation and thermal energy in an elastic solid is known as the thermoelastic effect. The first theoretical treatment of this phenomenon was published for a single material element by Weber and Thomson (Lord Kelvin) as early as 1855. Based on Kelvin's approach, it can be shown that for a linear elastic, homogenous material the rate of change of temperature (\dot{T}) is a function of the applied deformation in the form

$$\dot{T} = \frac{T_0}{\rho C_\epsilon} \frac{\partial \sigma_{ij}}{\partial T} \dot{\epsilon}_{ij} - \frac{\dot{Q}}{\rho C_\epsilon} \quad \text{for } i, j = 1, 2, 3 \quad (1)$$

where:

T	is the temperature
T_0	is the absolute (reference) temperature
C_ϵ	is the specific heat at constant strain
\dot{Q}	is the rate of heat production per unit volume
ρ	is the mass density
σ_{ij}	is the stress tensor
$\dot{\epsilon}_{ij}$	is the rate of change of the strain tensor

During TSA the test specimen is dynamically loaded at a frequency high enough so that the heat transfer term (\dot{Q}) can be neglected and hence, the relationship given in equation (1) can be assumed to occur adiabatically. The minimum frequency required is dependent on the thermal conductivity of the test material and stress gradients in the structure.

To develop equation (1) into a simple equation that can be applied experimentally it is necessary to express the stresses in terms of strains and temperature to derive $\partial \sigma_{ij} / \partial T$. For an isotropic, linear elastic material the constitutive stress-strain-temperature relationships can be expressed in terms of the Lamé constants in the form:

$$\sigma_{ij} = 2\mu\epsilon_{ij} + (\lambda\epsilon_{kk} - \beta\delta T)\delta_{ij} \quad (2)$$

for

$$\delta_{ij} = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases}$$

and

$$\beta = (3\lambda + 2\mu)\alpha$$

where ε_{kk} is the first strain invariant (i.e. $\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}$), δT is the change in temperature ($\delta T = T - T_0$) and α is the coefficient of linear thermal expansion. The Lamé constants: λ and μ , are functions of the Young's Modulus, E , and Poisson's ratio, ν , as follows

$$\mu = \frac{E}{2(1+\nu)} \quad \text{and} \quad \lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \quad (3)$$

The derivatives of stress with respect to T are obtained from equation (2) as follows:

$$\frac{\partial \sigma_{ij}}{\partial T} = 2 \frac{\partial \mu}{\partial T} \varepsilon_{ij} + \left(\frac{\partial \lambda}{\partial T} \varepsilon_{kk} - \frac{\partial \beta}{\partial T} \delta T - \beta \right) \delta_{ij} \quad (4)$$

Equation (4) contains **temperature derivatives of the material elastic properties**. For most engineering materials the variations of the elastic properties with temperature are practically zero at room temperature, so in this treatment they are neglected. For a stress induced temperature field, δT is in the order of mK and therefore the term $(\partial \beta / \partial T) \delta T$ will be negligible compared to β so this term is also neglected to give a simplified version of equation (4) as

$$\frac{\partial \sigma_{ij}}{\partial T} = -\beta \delta_{ij} \quad (5)$$

Substituting equation (5) into equation (1) and assuming **adiabatic conditions**, where $\dot{Q} = 0$, gives an expression for the rate of temperature change in terms of the material properties and the applied deformation, which can be written as

$$\dot{T} = -\frac{T_0 \beta}{\rho C_\varepsilon} \dot{\varepsilon}_{kk} \quad (6)$$

Expressing $\dot{\varepsilon}_{kk}$ in terms of stress using the relationship given in equation (2) yields:

$$\dot{T} = -\alpha \left[\frac{T_0}{\rho C_\varepsilon} + \frac{1-2\nu}{3\alpha^2 E} \right] \dot{\sigma}_{kk} \quad (7)$$

where $\dot{\sigma}_{kk}$ is the rate of change of the sum of the first stress invariant ($\sigma_{11} + \sigma_{22} + \sigma_{33}$).

The bracketed term in equation (7) is comprised mainly of material constants. However, T_0 may vary from test to test and it would be convenient to express equation (7) as a linear function of the material properties. To do this C_ε can be expressed in terms of the specific heat at constant pressure (C_p) using the relationship:

$$C_\varepsilon = C_p - \frac{3E\alpha^2 T_0}{\rho(1-2\nu)} \quad (8)$$

Therefore equation (7) can be re-written as

$$\dot{T} = -\frac{\alpha T_0}{\rho C_p} \dot{\sigma}_{kk} \quad (9)$$

Equation (9) directly relates the rate of change of temperature to the rate of change of stress in the specimen. Integrating this expression over a time period from the initial state to the final state of deformation provides a linear relationship between the change in temperature (ΔT) and the change in the first stress invariant ($\Delta\sigma_{kk}$) in the form

$$\Delta T = -KT_0\Delta\sigma_{kk} \quad (10)$$

where K is known as the thermoelastic constant and is a function of the material properties as follows

$$K = \frac{\alpha}{\rho C_p} \quad (11)$$

To yield the more familiar form of

$$\Delta T = \frac{\alpha T_0}{\rho C_p} \Delta(\sigma_1 + \sigma_2) \quad (12)$$

Equation (12) is the standard form of the thermoelastic equation for an isotropic, homogenous material, loaded elastically under adiabatic conditions.

Obtaining thermoelastic measurements from an infra-red system

In early work with the SPATE system the total radiant flux emitted from a surface was used to develop a working relationship for thermoelastic studies. It follows by differentiation of the standard Stefan-Boltzman relationship that the flux change, $\Delta\Phi$, resulting from a small change in the surface temperature, ΔT , is given by:

$$\Delta\Phi = 4eBT^3\Delta T \quad (13)$$

where e is the surface emissivity which is important to consider in TSA as it is probable that the surface will not behave like a blackbody (in the practical application of TSA it is usual that the surface is coated in a thin matt black paint layer to enhance and standardise the surface emissivity).

If the flux change is recorded by a linear detecting system, the detector voltage output (S) will be proportional to the change in temperature and therefore it follows from equation (13) that the change in the principal material stresses is given by:

$$S = -\frac{4R^*eB\rho C_p T^4}{\alpha} (\Delta\sigma_1 + \Delta\sigma_2) \quad (14)$$

where R^* is some detector response factor for the operating system.

Grouping the variables before the bracket on the right-hand side of equation (14) as those dependent on the material under test and the settings of the detector system, a calibration constant, A , is defined. The general thermoelastic relationship is therefore as follows:

$$AS = (\Delta\sigma_1 + \Delta\sigma_2) \quad (15)$$

This treatment neatly obviates the inaccuracies associated with using the simple Stefan-Boltzmann relationship. However the calibration constant has to be defined for every surface temperature as there is a very strong detector dependence on surface temperature particularly for the detectors that operate on the 2-5 μm wavelength range.

Temperature dependence of thermoelastic response

To deriving a temperature correction factor that could be realistically inserted into equation (15), an expression that is just a function of temperature is desirable. An approximate approach is suggested for detectors where the operating wavelength is less than λ_{max} . Here the response of the detector to temperature changes follows an approximate power law, i.e. $N_b \propto T^n$. Therefore it is possible to propose an equation that relates the surface temperature of a body to the total number of photons emitted over a particular wavelength range as follows:

$$N_{b_\lambda} = BT^n \quad (16)$$

where B is a constant that is dependent on the detector operating wavelength. The index n can be evaluated by determining N_{b_λ} from equation (8) by numerical integration over the wavelength of interest for a variety of temperatures. Taking logs of equation (16) will yield a linear form, i.e.:

$$\ln N_{b_\lambda} = \ln B + n \ln T \quad (17)$$

and n can be determined from the slope of a plot of $\ln N_{b_\lambda}$ against $\ln T$; B can be derived from the intercept of the plot.

For a specimen with an absolute temperature of 293 K λ_{max} occurs at 9.89 μm . Therefore as the power law is valid only for cases where the operating wavelength of the detector is less than λ_{max} , at around room temperature the relationship given by equation (17) is valid for the 2-5 μm range but not in the 8-12 μm range. A numerical integration of Planck's law over the 2-5 μm range for temperatures of 293 K to 323 K using MATLAB and application of the above procedure yielded $B = 6.322 \times 10^6 \text{ photons s}^{-1} \text{ m}^{-3} \text{ sr}^{-1} \text{ K}^{-10.47}$ and $n = 10.47$; the correlation coefficient for the curve fit was 0.99. Therefore a temperature correction factor can be introduced as $(T_0 / T)^{10.47}$. There is a huge difference between the standard approach, using the

photon detector law and calculating over the correct wavelength range. Hence equation (15) should be used with caution. It is much better to have an empirical radiometric calibration against a black body and apply equation (12) directly.

Derivation of the thermoelastic constant

Until recently the approach in TSA was to derive the calibration factor (A in equation (15)). The three principal calibration techniques used to derive a value for A , are:

- (i) Direct calibration, using properties of the infrared detector, system variables, specimen surface emissivity and the thermoelastic constant of the specimen material.
- (ii) Calibration against measured stress
- (iii) Calibration against calculated stress

Direct calibration (method (i)) is based on solving equation (14) and utilises both material and detector properties to theoretically derive a value for A . This method is not regarded as the most accurate. The second method utilises an independent measure of strain (typically an electrical resistance strain gauge). The sum of the principal stresses can be determined using the strain measurements and Hooke's law; however, values for the material's Young's modulus and Poisson's ratio are required. The third technique used for calibration requires a known stress field. Normally a specimen constructed from the material in question is loaded in simple tension. (A beam in four point bending or a disc in two point diametral compression are recommended arrangements.) The stress can be calculated theoretically using the cross-sectional area and the applied load. Equation (15) is simplified to find a value for A . This method has least potential for error sources.

Identical approaches can be used to find the thermoelastic constant K . Here equation (12) is used, a known stress is applied to a specimen ΔT and T are obtained from the measurement and therefore K is obtained.

Non-adiabatic conditions

Non-adiabatic conditions exist when the rate of heat flux per unit volume (\dot{Q}) is greater than zero. It is impossible to achieve fully adiabatic conditions experimentally; however, for the purposes of TSA a *pseudo-adiabatic* state may be achieved, where there is no measurable attenuation of the thermoelastic signal due to heat transfer. To understand the limitations of the pseudo-adiabatic assumption it is useful to use the following expression:

$$\dot{T} = -\frac{\alpha T_0}{\rho C_p} \dot{\sigma}_{kk} - \frac{\dot{Q}}{\rho C_\epsilon} \quad (18)$$

The magnitude of $\dot{\sigma}_{kk}$ can be amplified by increasing the loading frequency and thereby making it much greater than the \dot{Q} term. This is usually the approach taken to obtain a pseudo-adiabatic state; with a loading frequency of 10 to 20 Hz being sufficient for most applications.

\dot{Q} is a function of the thermal conductivity of the material and the stress induced thermal gradients in the component, and can be expressed in the form

$$\dot{Q} = k\nabla^2 T \quad (19)$$

where k is the material thermal conductivity and $\nabla^2 T$ is the thermal gradient defined as:

$$\nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \quad (20)$$

Analysis of equation shows that for a specimen with a uniform stress field or a low k value the \dot{Q} term will be small. In both cases a pseudo-adiabatic state is achieved at low loading frequencies. However, most engineering components have stress gradients and are made from metals that have a high thermal conductivity.

FEA may be used to analyse non-adiabatic affects at loading frequencies that can not be achieved experimentally. It may be used to determine the loading frequency where \dot{Q} can be neglected for a given stress field and a given set of material properties. In close proximity of stress raisers attenuation of the thermoelastic signal is an important consideration, even at loading frequencies of 20 Hz and above. Often the attenuation effects are localised and TSA can still be used to obtain quantitative data from the majority of the specimen. This is verified by the large number of publications that have successfully used TSA to obtain quantitative data at loading frequencies less than 15 Hz.

Non-adiabatic behaviour can be investigated experimentally by examining the thermoelastic response over a range of frequencies. If the response is independent of frequency it can be assumed that a pseudo-adiabatic state has been achieved. The phase also provides a convenient tool for assessing non-adiabatic behaviour. Heat conduction in areas of high stress gradient results in change in the thermoelastic signal magnitude and a phase shift. Discontinuities in the phase image can be used to identify non-adiabatic regions in the specimen. This is evident in Figure 1 where areas of high out-of-phase signal are reported at the edges of the hole in the bar where the stress gradient is greatest.

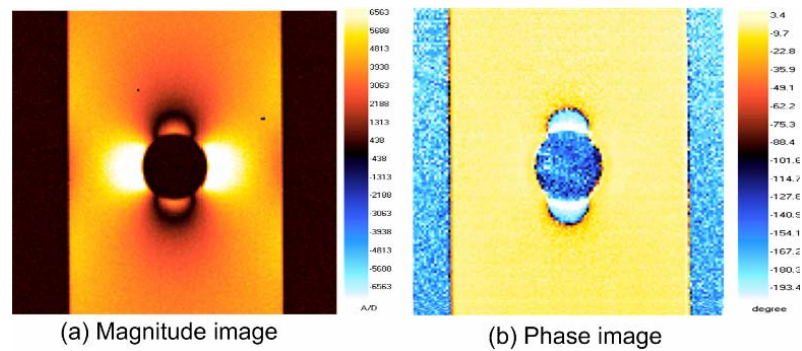


Figure 1 TSA data from hole in a plate specimen

The resolution of the detection system is also a consideration when obtaining pseudo-adiabatic conditions. TSA of small scale structures requires a detection system with a high spatial resolution to ensure that an adequate number of measurements are obtained across the structure to build an accurate representation of the signal distribution. However, at high resolutions each detector element is projected over a very small area on the test specimen. The average temperature in the projected area is used to derive the thermoelastic signal. The smaller the measurement area the greater effect the \dot{Q} term will have on the thermoelastic measurements. At high detector resolutions acute stress gradients are not averaged out over large measurement areas and therefore the sensitivity of the signal to non-adiabatic effects is increased.

Paint coatings

It is standard practice when taking thermographic measurements to coat the specimen surface with a thin layer of matt black paint to improve emissivity and ensure the response is uniform. However, this coating is thermally inert in comparison with the specimen and may in certain instances give rise to non-adiabatic effects. Thermal lag is caused by the insulating effect of the paint coating. Even with no heat loss to the environment, a surface coating will still have a capacitance and resistance associated with it; this will result in a temperature drop across it. In short, an increase in paint coating thickness will result in a smaller ΔT at the paint surface than at the substrate surface. This is significant for small scale specimens where it can be difficult to apply an even paint coating, especially in areas where there are sharp changes in section. In general, most practitioners will recommend a minimal amount of paint. Two short passes of paint supplied using an aerosol is deemed suitable for most applications.

Thermal drag-down is a phenomenon relating to both the paint thickness and the loading frequency. Since no heat is being generated by the coating, its temperature can only change due to the heat transferred to it from the substrate. As the loading frequency increases, the spatial wavelength decreases; as a result there is less heat input into the coating. In brief, the heat will not flow into the surface coating sufficiently quickly to maintain the temperature change for accurate thermoelastic measurements to be recorded.

Temperature dependence of the material elastic properties

For the small changes in temperatures associated with the thermoelastic effect, K , the thermoelastic constant, is assumed to be a material constant that is independent of the stress field. However, K is a function of the coefficient of linear expansion, α that has been shown to be stress dependent as follows:

$$\left(\frac{\partial \alpha}{\partial \sigma}\right)_T = \frac{\partial}{\partial \sigma} \left[\left(\frac{\partial \varepsilon}{\partial T}\right)_\sigma \right]_T = \frac{\partial}{\partial T} \left[\left(\frac{\partial \varepsilon}{\partial \sigma}\right)_T \right]_\sigma \quad (21)$$

Therefore if $E = (\partial \sigma / \partial \varepsilon)_T$, differentiating by parts gives

$$\left(\frac{\partial \alpha}{\partial \sigma}\right)_T = -\frac{1}{E^2} \left(\frac{\partial E}{\partial T}\right)_\sigma \quad (22)$$

Equation (22) shows that α will be stress dependent for materials whose elastic properties are sensitive to changes in temperature. This also implies that the thermoelastic response of such materials will be dependent on the mean stress of the applied stress cycle and can not be assumed to be constant.

The theoretical approach presented previous assumes that the dependence is a higher order effect and can be neglected for most engineering materials. Although this hypothesis has been shown to be valid for most applications, it may not be valid for some materials, particularly those with high temperature sensitivity used in components where a high mean stress is applied and the following equation must be used:

$$\dot{T} = \frac{T_0}{\rho C_\varepsilon} \left[\left(-\beta - \frac{\partial \beta}{\partial T} \delta T + \frac{\partial \lambda}{\partial T} \varepsilon_{kk} \right) \delta_{ij} + 2 \frac{\partial \mu}{\partial T} \varepsilon_{ij} \right] \dot{\varepsilon}_{ij} - \frac{\dot{Q}}{\rho C_\varepsilon} \quad (23)$$

The $(\partial \beta / \partial T) \delta T$ term will be negligible compared to β and can be neglected. However, for some materials the terms $(\partial \mu / \partial T) \varepsilon_{ij}$ and $(\partial \lambda / \partial T) \varepsilon_{kk}$ can be of a significant order. Therefore retaining these terms and omitting higher order components allows equation (23) to be written in terms of stresses for adiabatic conditions as follows:

$$\dot{T} = \frac{T_0}{\rho C_p} \left\{ - \left(\alpha + \left(\frac{\nu}{E^2} \frac{\partial E}{\partial T} - \frac{1}{E} \frac{\partial \nu}{\partial T} \right) \sigma_{kk} \right) \dot{\sigma}_{kk} + \left(\frac{(1+\nu)}{E^2} \frac{\partial E}{\partial T} - \frac{1}{E} \frac{\partial \nu}{\partial T} \right) \sigma_{ij} \dot{\sigma}_{ij} \right\} \quad (24)$$

Equation (24) is known as the revised version of the thermoelastic equation where the rate of temperature change is both a function of the stresses and their rate of change. It should be noted that C_p is used in equation (24). The higher order terms that constitute the difference between C_ε and C_p are neglected to allow equation (24) to be presented in a concise form. The difference between C_ε and C_p can be calculated using equation (8) and is negligible for most metals and hence C_p is used here for consistency.

To illustrate more clearly the effects of the stresses in equation (24) a uniaxial stress system is considered, where $\sigma_{11} = \sigma_{kk}$ and $\sigma_{22} = \sigma_{33} = \sigma_{12} = \sigma_{23} = \sigma_{13} = 0$. Under these conditions equation (24) can be written as follows

$$\dot{T} = -\frac{T_0}{\rho C_p} \left(\alpha - \frac{1}{E^2} \frac{dE}{dT} \sigma_{11} \right) \dot{\sigma}_{11} \quad (25)$$

Equation (25) shows that for the revised treatment the rate of temperature change is a function of the applied stress and its rate of change. It is now possible to integrate equation (25) over the time period corresponding to the deformation in a similar manner to the approach with equation (9). However, in a simple deformation process it is difficult to see what value should be used for the σ_{11} , i.e. a static stress, which is independent of time. In a cyclic load, as with that used in TSA, σ_{11} could be regarded as the mean stress of the cycle, σ_m , as this does not vary with time, and therefore equation (25) can be integrated to give:

$$\Delta T = -\frac{T_0}{\rho C_\epsilon} \left(\alpha - \frac{1}{E^2} \frac{dE}{dT} \sigma_m \right) \Delta \sigma_{11} \quad (26)$$

So that

$$\Delta T = -K^* \Delta \sigma_{11} \quad (27)$$

where K^* is the revised version of the thermoelastic constant that is a function of the Young's modulus and the mean stress and can be written as

$$K^* = \frac{1}{\rho C_p} \left(\alpha - \frac{1}{E^2} \frac{dE}{dT} \sigma_m \right) \quad (28)$$

Equation (28) provides an expression for the thermoelastic constant that takes the stress dependence of α into account by incorporating the quantity $(1/E^2)dE/dT$ (see equation (26)) and the σ_m term. If the material's Young's modulus is high and the dE/dT term is small then the mean stress will have a negligible effect on the thermoelastic response. The stress dependence of the thermoelastic constant for a variety of materials is given in Table 1. The percentage difference between K^* values calculated at zero stress and at the yield stress was evaluated for each material. The greatest dependence (40%) was calculated for Ti-6Al-4V, with values of 10% and 14% obtained for Al-2024 and 4340 steel respectively. Analysis of the material properties given in Table 1 shows that relative to the other materials Ti-6Al-4V has a relatively low E value combined with a high dE/dT and yield strength values.

Table 1 Material properties for three high strength alloys

Material	E (MPa)	dE/dT (MPa/K)	σ_y (MPa)
4340 steel	210×10^3	-56.7	304
Al-2024	72×10^3	-36.0	197
Ti-6Al-4V	120×10^3	-61.8	430

The practical significance of the mean stress effect for the materials given in Table Table 1 can be established by calculating the percentage of the material's yield stress, required to introduce an error of 5% by neglecting the mean stress dependence was examined. Calculations were carried out for steel, an aluminium alloy and a titanium alloy with values of 29%, 17% and 5% obtained respectively. In practical terms this analysis shows that, at least in the case of steel and aluminium, the mean stress effect will be effectively hidden within the ambient noise of the measurements (approximately 5%). The effect is shown to be significant in the case of the titanium alloys where a mean stress that is just 5% of the yield stress is required to introduce an error of 5% in the thermoelastic measurements.